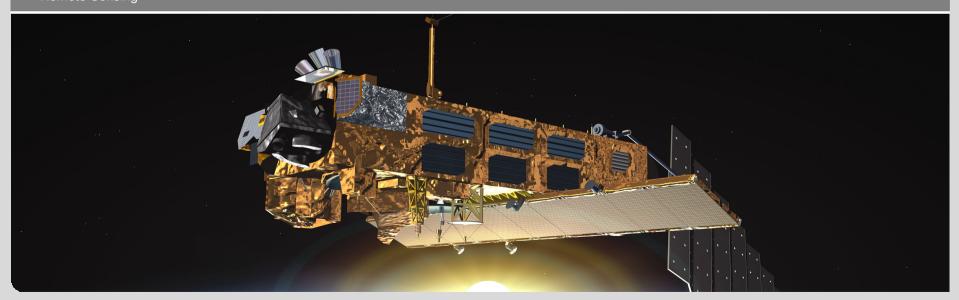


The MIPAS ozone record since 2002

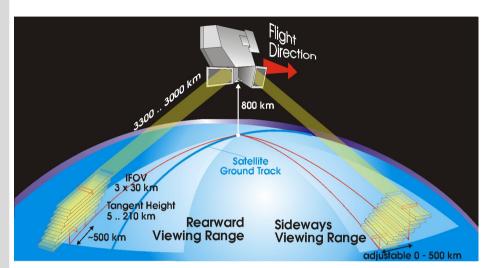
Thomas von Clarmann and the MIPAS-Envisat teams at IMK, KIT Karlsruhe and IAA, CSIC, Granada, Spain

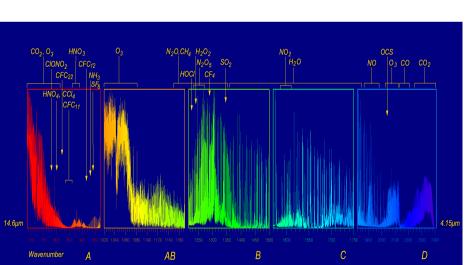
Institute for Meteorology and Climate Research – Atmospheric Trace Gases and Remote Sensing



MIPAS measurement principle





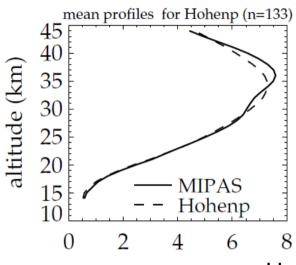


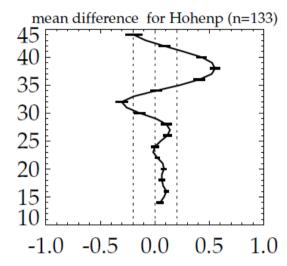
- IR limb emission spectrometer
- Measures day and night
- Altitude range 6 to 70 km (170 km)
- Pole-to-pole, > 1000 profiles/day
- So far 30 trace species, temperature and cloud composition
- 2002 2004: full spectral res., vertical resolution 3.5 6 km
- Since 2005: reduced spectral resolution, vertical resolution improved (2 - ... km)
- Non-operational scientific analysis of MIPAS data at IMK/IAA

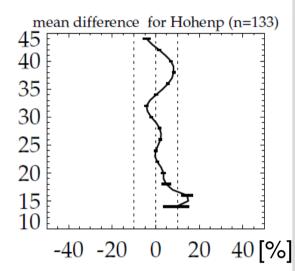
Validation of full-resolution MIPAS ozone profiles (examples)



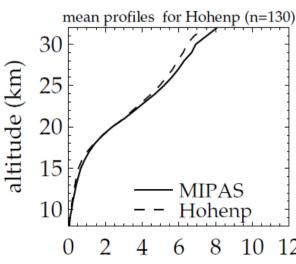
Hohenpeissenberg LIDAR measurements

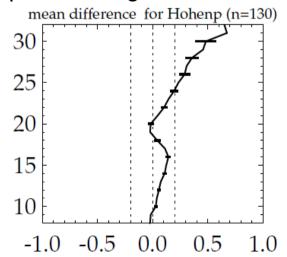


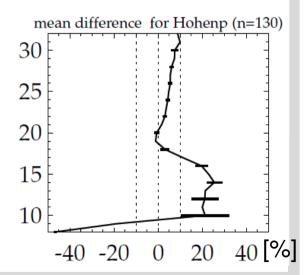




Hohenpeissenberg ozone sondes

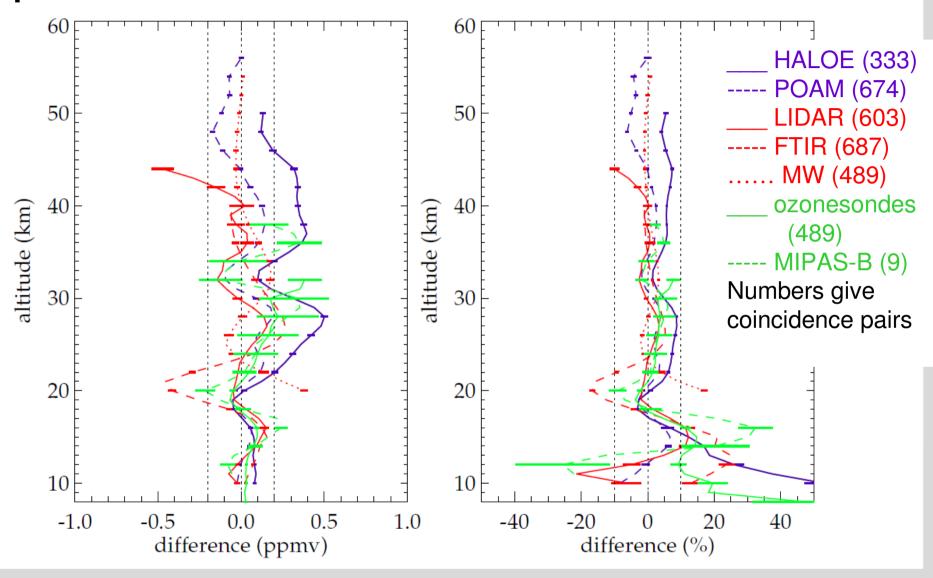






Validation of full-resolution MIPAS ozone profiles cntd.





MIPAS optimized-resolution retrievals (2005 – ...)

Kerlsruhe Institute of Technologi

Spectral resolution: 0.0625 cm⁻¹ unapodized

Table 7. O₃ retrieval error budget.

Height km	Noise ppbv (%)	Pointing ppbv (%)	Gain ppbv (%)	T ppbv (%)	Precision ppbv (%)	ILS ppbv (%)	Spectroscopy ppbv (%)	Total error ppbv (%)
50	75.0 (2.1)	180.0 (5.1)	25.0 (0.7)	76.0 (2.1)	210.8 (5.9)	2.5 (0.1)	230.0 (6.5)	312.0 (8.8)
40	50.0 (0.9)	210.0 (3.6)	120.0 (2.1)	130.0 (2.2)	279.1 (4.8)	0.3 (0.0)	650.0 (11.1)	707.4 (12.1)
35	73.0 (1.3)	170.0 (3.1)	95.0 (1.7)	78.0 (1.4)	222.1 (4.0)	85.0 (1.5)	620.0 (11.2)	664.1 (12.0)
30	67.0 (1.1)	200.0 (3.3)	130.0 (2.2)	67.0 (1.1)	256.7 (4.3)	170.0 (2.8)	720.0 (12.1)	783.1 (13.1)
25	60.0 (1.0)	220.0 (3.7)	130.0 (2.2)	85.0 (1.4)	275.9 (4.7)	130.0 (2.2)	700.0 (11.9)	763.6 (12.9)
20	46.0 (1.3)	51.0 (1.4)	120.0 (3.3)	22.0 (0.6)	140.0 (3.8)	89.0 (2.4)	410.0 (11.2)	442.3 (12.1)
15	33.0 (1.9)	1.7(0.1)	62.0 (3.6)	9.8 (0.6)	70.9 (4.1)	28.0 (1.6)	150.0 (8.6)	168.3 (9.6)
10	28.0 (7.0)	38.0 (9.5)	13.0 (3.2)	13.0 (3.2)	50.7 (12.6)	22.0 (5.5)	40.0 (10.0)	68.2 (17.0)

Vertical resolution: 2.4km @ 20km to

3.5km @ 50km

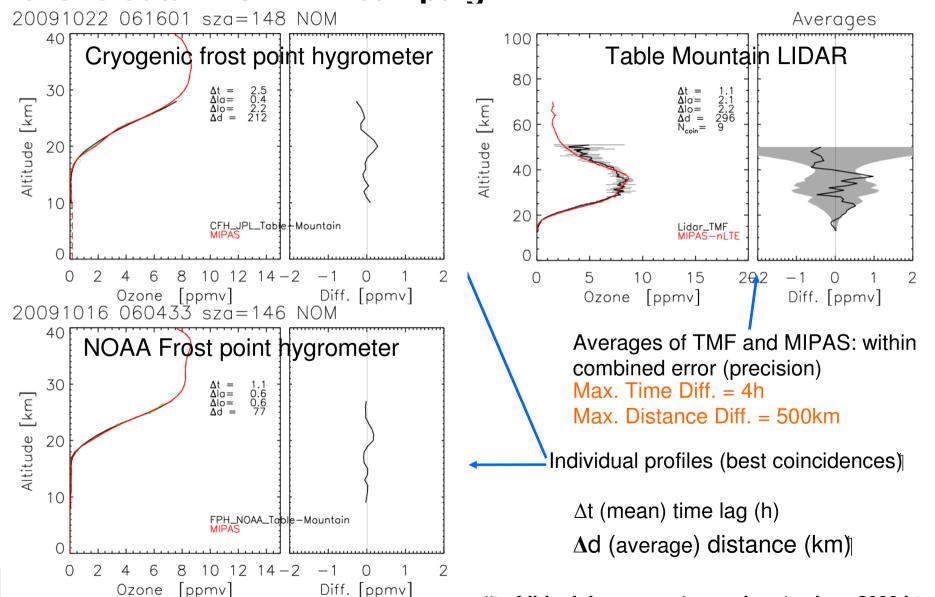
Horizontal resolution: 250km @ 10km to

400km @ 40 km

Validation of optimized-resolution MIPAS ozone data: MOHAVE campaign

25 January 2011



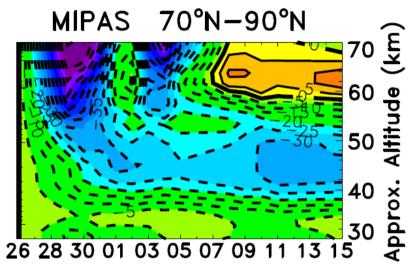


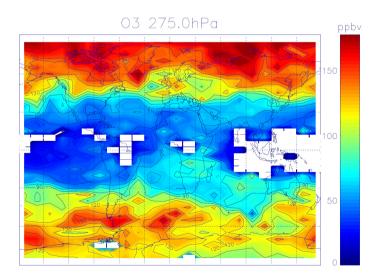
nttp://tmf-lidar.jpl.nasa.gov/campaigns/mohave2009.htm

We see episodes ...:



Ozone reduction after SPE (Lopez-Puertas et al, 2005; Jackman et al., 2008



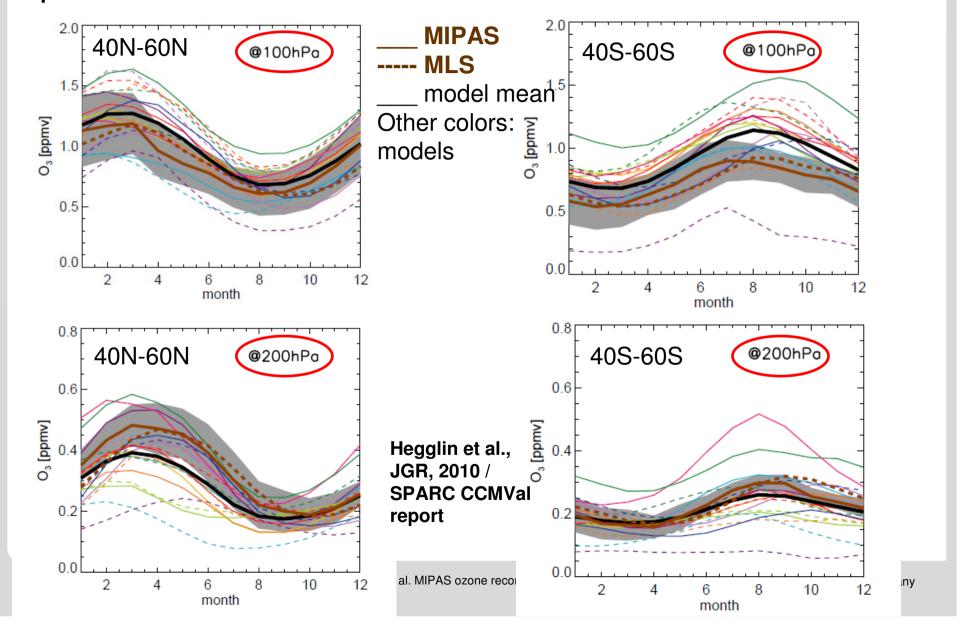


Biomass burning, v. Clarmann et al. 2007,

Use of MIPAS O3 in SPARC-CCMVal activities

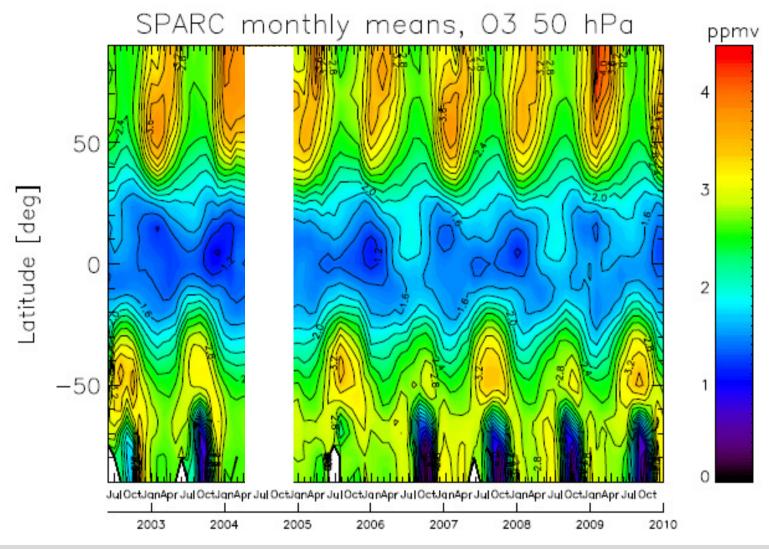
Validation of phase and amplitude of the seasonal cycle of ozone in the extratropical UTLS





The MIPAS data record since 2002





How to cope with different characteristics?



- Rough estimate of the bias (not even the sign must be known, only the order of size of the bias)
- Build covariance matrix of data errors, where a fully correlated error block is added to one subset of the data to describe the bias uncertainty.
- Calculate linear regression under consideration of the full covariance matrix

$$b = \frac{x^{T} S_{y}^{-1} y - \frac{x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} y}{e^{T} S_{y}^{-1} e}}{x^{T} S_{y}^{-1} x - \frac{x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}{e^{T} S_{y}^{-1} e}}$$

$$= \frac{x^{T} S_{y}^{-1} y - e^{T} S_{y}^{-1} x}{x^{T} S_{y}^{-1} x - \frac{x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}{e^{T} S_{y}^{-1} e}}$$

$$= \frac{x^{T} S_{y}^{-1} y e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} y}{e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}$$

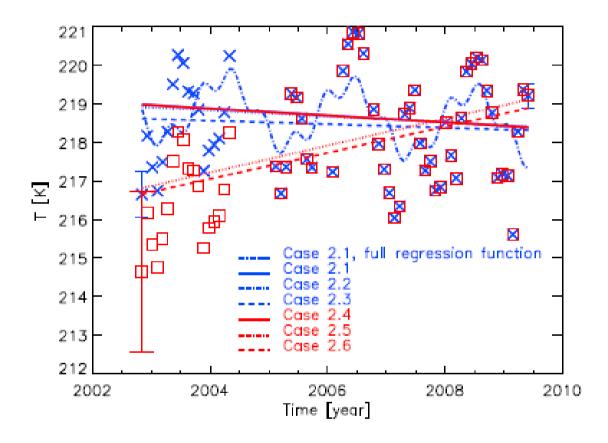
$$= \frac{e^{T} S_{y}^{-1} y - e^{T} S_{y}^{-1} x \frac{x^{T} S_{y}^{-1} y e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} y}{e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}$$

$$= \frac{e^{T} S_{y}^{-1} y x^{T} S_{y}^{-1} x - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} y}{e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}.$$

T. von Clarmann et al., ACP, 2010

Karlaruhe Institute of Technology

It works!



... but treatment of a superimposed periodic function requires additional sophistication

Conclusions:



KIT, IMK-ASF, Karlsruhe, Germany

- MIPAS ozone records (global, altitude resolved) are available since 2002.
- These data sets have been validated against various independent measurements.
- Accurate data characterization (error covariance matrices, horizontal and vertical averaging kernels) are available.
- The measurement mode and thus data characteristics have changed in 2004.
- A method has been developed to cope with such data inconsistencies, data inhomogeneities, clustered data etc...

This tool may also be useful to infer trends from multiple data sources.

$$\sigma_{
m mean} = \sqrt{\sigma^2 igg(rac{1+(M-1)ar{r}_{
m inter}}{M}}$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{1}{M} \end{pmatrix}$$

$$S_{y} = S_{\text{noise}} + \begin{pmatrix} \left(\begin{array}{c} \text{bias} \frac{2}{2,1} & \dots & bias \frac{2}{2,1} \\ \vdots & \ddots & \vdots \\ \text{bias} \frac{2}{2,1} & \dots & bias \frac{2}{2,1} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \end{array} \right) \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \end{pmatrix}, (15)$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{1}{M} \end{pmatrix}$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{1}{M} \end{pmatrix}$$

$$S_{y} = S_{\text{noise}} + \begin{pmatrix} \left(\begin{array}{c} \text{bias} \frac{2}{2,1} & \dots & bias \frac{2}{2,1} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \end{array} \right) \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \end{pmatrix}, (15)$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{1}{M} \end{pmatrix}$$

$$S_{y} = S_{\text{noise}} + \begin{pmatrix} \left(\begin{array}{c} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \\ \frac{\partial d}{\partial y} \end{pmatrix} \right) S_{y} \begin{pmatrix} T^{-1} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial d}{\partial y} \end{pmatrix} \end{pmatrix}^{T} \quad \text{and}$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{\text{inter}} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} T^{-1} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial d}{\partial y} \end{pmatrix} \end{pmatrix}^{T} \quad \text{and}$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{\text{inter}} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} T^{-1} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial d}{\partial y} \end{pmatrix} \end{pmatrix}^{T} \quad \text{and}$$

$$\sigma_{\text{mean,inter}}^{2} = (\frac{1}{M}, \dots, \frac{1}{M}) S_{\text{inter}} \begin{pmatrix} \frac{1}{M} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{\text{inter}} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \vdots \\ \frac{\partial d}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac{\partial q}{\partial y} \\ \frac{\partial q}{\partial y} \end{pmatrix} S_{y} \begin{pmatrix} \frac$$

(11)

$$\begin{split} \mathbf{S}_{a,b,c,d} &= \begin{pmatrix} \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial y} \\ \frac{\partial c}{\partial y} \\ \frac{\partial d}{\partial y} \end{pmatrix} \mathbf{S}_{y} \begin{pmatrix} \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial y} \\ \frac{\partial c}{\partial y} \\ \frac{\partial c}{\partial y} \end{pmatrix} \mathbf{S}_{y} \begin{pmatrix} \frac{\partial a}{\partial y} \\ \frac{\partial c}{\partial y} \\ \frac{\partial c}{\partial y} \end{pmatrix} &= e^{T} \mathbf{S}_{y}^{-1} a e + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} a e + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y = e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} y - e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y}^{-1} b x; \\ &= e^{T} \mathbf{S}_{y}^{-1} b x + e^{T} \mathbf{S}_{y$$

 $d_{i+1}\operatorname{diag}\left((a_{i}e + b_{i}x)v_{\cos}^{T}\right)$;

$$\tilde{a} = \frac{\sum y_n}{N} - \tilde{b} \frac{\sum x_n}{N},$$

$$\tilde{b} = \frac{N \sum x_n y_n - \sum x_n \sum y_n}{N \sum x_n^2 - (\sum x_n)^2}$$

The uncertainty of the slope b is:

$$\sigma_b^2 = \left(\frac{\partial b}{\partial y}\right) S_y \left(\frac{\partial b}{\partial y}\right)^T$$

$$= \left(\frac{e^T S_y^{-1} e x^T S_y^{-1} - x^T S_y^{-1} e e^T S_y^{-1}}{x^T S_y^{-1} x e^T S_y^{-1} e - x^T S_y^{-1} e e^T S_y^{-1} x}\right).$$

$$S_{y} \left(\frac{e^{T} S_{y}^{-1} e x^{T} S_{y}^{-1} - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1}}{x^{T} S_{y}^{-1} x e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x} \right)^{T}$$

where $\left(\frac{\partial b}{\partial y}\right) = \left(\frac{\partial b}{\partial y_1} \dots \frac{\partial b}{\partial y_N}\right)$. The uncertainty of a is estimated accordingly: $\frac{\partial \chi^2}{\partial b}$

$$\begin{split} &\sigma_{a}^{2} = \left(\frac{\partial a}{\partial y}\right) S_{y} \left(\frac{\partial a}{\partial y}\right)^{T} & -2x^{T} S_{y}^{-1} (y(x) - ae - bx - cv_{\sin} - dv_{\cos}) = \\ &= \left(\frac{x^{T} S_{y}^{-1} x e^{T} S_{y}^{-1} - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1}}{e^{T} S_{y}^{-1} e x^{T} S_{y}^{-1} x - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}\right) \cdot & \frac{\partial \chi^{2}}{\partial c} = \\ &S_{y} \left(\frac{x^{T} S_{y}^{-1} x e^{T} S_{y}^{-1} - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1}}{e^{T} S_{y}^{-1} e x^{T} S_{y}^{-1} x - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}\right)^{T} & \frac{\partial \chi^{2}}{\partial c} = \\ &S_{y} \left(\frac{x^{T} S_{y}^{-1} x e^{T} S_{y}^{-1} - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}{e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}\right)^{T} & \frac{\partial \chi^{2}}{\partial c} = \\ &-2v_{\cos}^{T} S_{y}^{-1} (y(x) - ae - bx - cv_{\sin} - dv_{\cos}) = 0 \\ &\frac{\partial \chi^{2}}{\partial d} = \\ &-2v_{\cos}^{T} S_{y}^{-1} (y(x) - ae - bx - cv_{\sin} - dv_{\cos}) = 0, \end{split}$$

Thank you for your patience!

$$= \left(\frac{e^{T}S_{y}^{-1}ex^{T}S_{y}^{-1} - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}}{x^{T}S_{y}^{-1}xe^{T}S_{y}^{-1}e - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}\right).$$

$$Sy\left(\frac{e^{T}S_{y}^{-1}xe^{T}S_{y}^{-1} - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}}{x^{T}S_{y}^{-1}xe^{T}S_{y}^{-1}e - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}\right)^{T},$$

$$Sy\left(\frac{e^{T}S_{y}^{-1}xe^{T}S_{y}^{-1} - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}{x^{T}S_{y}^{-1}xe^{T}S_{y}^{-1}e - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}\right)^{T},$$

$$Sy\left(\frac{e^{T}S_{y}^{-1}xe^{T}S_{y}^{-1} - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}{x^{T}S_{y}^{-1}xe^{T}S_{y}^{-1}e - x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}x}\right)$$

$$C_{i+1}\text{diag}\left((a_{i}e + b_{i}x)v_{\text{sin}}^{T}\right) - c_{i+1}^{T}S_{y}^{-1}e^{T}S_{y$$

e uncertainty of
$$\frac{\partial \chi^2}{\partial b} = -2x^T S_y^{-1} (y(x) - ae - bx - cv_{sin} - dv_{cos}) =$$

$$\begin{aligned} \frac{\partial \chi^2}{\partial c} &= \\ &- 2v_{\sin}^T \mathbf{S}_y^{-1} (y(x) - ae - bx - cv_{\sin} - dv_{\cos}) = 0 \end{aligned}$$

$$\frac{\partial \chi^2}{\partial d} = -2v_{\cos}^T S_y^{-1} (y(x) - ae - bx - cv_{\sin} - dv_{\cos}) = 0,$$

$$\mathbf{S}_{a,b,c,d} = \begin{pmatrix} \frac{\partial a}{\partial \mathbf{y}} \\ \frac{\partial b}{\partial \mathbf{y}} \\ \frac{\partial c}{\partial \mathbf{y}} \end{pmatrix} \mathbf{S}_{\mathbf{y}} \begin{pmatrix} \frac{\partial a}{\partial \mathbf{y}} \\ \frac{\partial b}{\partial \mathbf{y}} \\ \frac{\partial c}{\partial \mathbf{y}} \\ \frac{\partial c}{\partial \mathbf{y}} \\ \frac{\partial c}{\partial \mathbf{y}} \\ \frac{\partial c}{\partial \mathbf{y}} \end{pmatrix}^{T} \qquad \qquad \frac{\partial \chi^{2}}{\partial a} = -2e^{T}\mathbf{S}_{\mathbf{y}}^{-1}(\mathbf{y} - a\mathbf{e} - b\mathbf{x}) = 0;$$

$$e^{T}\mathbf{S}_{\mathbf{y}}^{-1}\mathbf{y} = e^{T}\mathbf{S}_{\mathbf{y}}^{-1}a\mathbf{e} + e^{T}\mathbf{S}_{\mathbf{y}}^{-1}b\mathbf{x};$$

$$a = \frac{e^{T}\mathbf{S}_{\mathbf{y}}^{-1}\mathbf{y} - e^{T}\mathbf{S}_{\mathbf{y}}^{-1}b\mathbf{x}}{e^{T}\mathbf{S}_{\mathbf{y}}^{-1}e}$$

$$\begin{split} \frac{\partial \chi^2}{\partial b} &= -2x^T S_y^{-1} (y - ae - bx) = 0 \\ x^T S_y^{-1} y - x^T S_y^{-1} ae - x^T S_y^{-1} bx &= 0 \\ x^T S_y^{-1} y &= x^T S_y^{-1} bx + x^T S_y^{-1} ae. \end{split}$$

$$x^{T}S_{y}^{-1}y = x^{T}S_{y}^{-1}bx + x^{T}S_{y}^{-1}ae$$

$$= x^{T}S_{y}^{-1}bx + x^{T}S_{y}^{-1}e\frac{e^{T}S_{y}^{-1}y - e^{T}S_{y}^{-1}bx}{e^{T}S_{y}^{-1}}$$

This can be rearranged as

$$x^{T}S_{y}^{-1}bx - \frac{x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}bx}{e^{T}S_{y}^{-1}e} =$$

$$x^{T}S_{y}^{-1}y - \frac{x^{T}S_{y}^{-1}ee^{T}S_{y}^{-1}y}{e^{T}S_{y}^{-1}e}$$
(8)

and finally solved to give b:

$$\begin{array}{ll}
 & \frac{x^T S_y^{-1} e e^T S_y^{-1} y}{e^T S_y^{-1} e} \\
 & \frac{x^T S_y^{-1} e e^T S_y^{-1} x}{e^T S_y^{-1} e} \\
 & \frac{e^T S_y^{-1} e e^T S_y^{-1} x}{e^T S_y^{-1} e} \\
 & \frac{e^T S_y^{-1} e e^T S_y^{-1} y}{e^T e^T S_y^{-1} e^T S_y^{-1} x}.
\end{array}$$
(9)

$$\frac{e^T S_y^{-1} x b}{e^T S_y^{-1} e}$$
(10)

$$= \frac{e^{T} S_{y}^{-1} y - e^{T} S_{y}^{-1} x \frac{x^{T} S_{y}^{-1} y e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} y}{e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}$$

$$= \frac{e^{T} S_{y}^{-1} y x^{T} S_{y}^{-1} x - e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x}{e^{T} S_{y}^{-1} e - x^{T} S_{y}^{-1} e e^{T} S_{y}^{-1} x x^{T} S_{y}^{-1} e}.$$